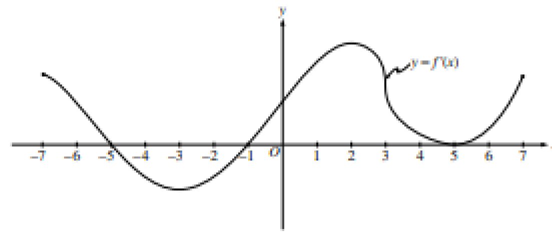
FRQ Test Saturday Review Problems with Answers

1. Function Analysis (No Calculator)

A.

The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.



(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

(d) At what value of x, for $-7 \le x \le 7$, does f attain its absolute maximum? Justify your answer.

(a) $x = -1$

$f'(x)$ changes from negative to positive at $x = -1$

$$2\begin{cases} 1: \text{ answer} \\ 1: \text{ justification} \end{cases}$$

(b) $x = -5$

$f'(x)$ changes from positive to negative at $x = -5$

$$2\begin{cases} 1: \text{ answer} \\ 1: \text{ justification} \end{cases}$$

(c) $f''(x)$ exists and f' is decreasing on the intervals $(-7,-3)$, $(2,3)$, and $(3,5)$

$$2\begin{cases} 1: \ (-7,-3) \\ 1: \ (2,3) \cup (3,5) \end{cases}$$

(d) $x = 7$

The absolute maximum must occur at $x = -5$ or at an endpoint.

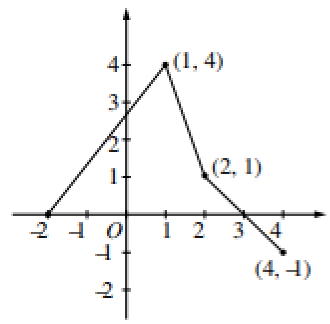
$f(-5) > f(-7)$ because f is increasing on $(-7,-5)$

The graph of f' shows that the magnitude of the negative change in f from $x = -5$ to $x = -1$ is smaller than the positive change in f from $x = -1$ to $x = 7$. Therefore the net change in f is positive from $x = -5$ to $x = 7$, and $f(7) > f(-5)$. So $f(7)$ is the absolute maximum.

$$3\begin{cases} 1: \text{ answer} \\ 1: \text{ identifies } x = -5 \text{ and } x = 7 \\ \quad \text{as candidates} \\ \quad - \text{ or } - \\ \quad \text{indicates that the graph of } f \\ \quad \text{increases, decreases, then increases} \\ 1: \text{ justifies } f(7) > f(-5) \end{cases}$$

B

The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.



(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

(a) $g(4) = \int_1^4 f(t)\,dt = \frac{3}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\frac{1}{2}(12) = -6$

$\mathbf{2}\begin{cases} 1\text{: } g(4) \\ 1\text{: } g(-2) \end{cases}$

(b) $g'(1) = f(1) = 4$

1: answer

(c) g is increasing on $[-2, 3]$ and decreasing on $[3, 4]$.

Therefore, g has absolute minimum at an endpoint of $[-2, 4]$.

Since $g(-2) = -6$ and $g(4) = \frac{5}{2}$,

the absolute minimum value is -6.

$\mathbf{3}\begin{cases} 1\text{: interior analysis} \\ 1\text{: endpoint analysis} \\ 1\text{: answer} \end{cases}$

(d) One; $x = 1$

On $(-2, 1)$, $g''(x) = f'(x) > 0$

On $(1, 2)$, $g''(x) = f'(x) < 0$

On $(2, 4)$, $g''(x) = f'(x) < 0$

Therefore $(1, g(1))$ is a point of inflection and $(2, g(2))$ is not.

$\mathbf{3}\begin{cases} 1\text{: choice of } x = 1 \text{ only} \\ 1\text{: show } (1, g(1)) \text{ is a point of inflection} \\ 1\text{: show } (2, g(2)) \text{ is not a point of inflection} \end{cases}$

2. Area Volume (Calculator Allowed)

Let R be the region bounded by the x–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.

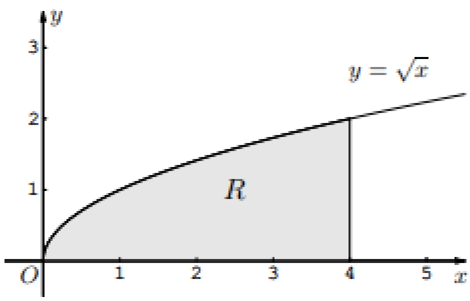
(a) Find the area of the region R.

(b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.

(c) Find the volume of the solid generated when R is revolved about the x–axis.

(d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.

(a)



$$A = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3} \quad \text{or} \quad 5.333$$

$2\begin{cases} 1: & A = \int_0^4 \sqrt{x}\,dx \\ 1: & \text{answer} \end{cases}$

(b) $\displaystyle\int_0^h \sqrt{x}\,dx = \frac{8}{3}$ –or– $\displaystyle\int_0^h \sqrt{x}\,dx = \int_h^4 \sqrt{x}\,dx$

$\displaystyle\frac{2}{3}h^{3/2} = \frac{8}{3}$ –or– $\displaystyle\frac{2}{3}h^{3/2} = \frac{16}{3} - \frac{2}{3}h^{3/2}$

$h = \sqrt[3]{16}$ or 2.520 or 2.519

$2\begin{cases} 1: & \text{equation in } h \\ 1: & \text{answer} \end{cases}$

(c) $\displaystyle V = \pi\int_0^4 (\sqrt{x})^2\,dx = \pi\frac{x^2}{2}\Big|_0^4 = 8\pi$

or 25.133 or 25.132

$3\begin{cases} 1: & \text{limits and constant} \\ 1: & \text{integrand} \\ 1: & \text{answer} \end{cases}$

(d) $\displaystyle\pi\int_0^k (\sqrt{x})^2\,dx = 4\pi$ –or– $\displaystyle\pi\int_0^k (\sqrt{x})^2\,dx = \pi\int_k^4 (\sqrt{x})^2\,dx$

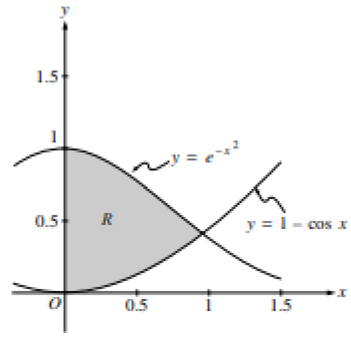
$\displaystyle\pi\frac{k^2}{2} = 4\pi$ –or– $\displaystyle\pi\frac{k^2}{2} = 8\pi - \pi\frac{k^2}{2}$

$k = \sqrt{8}$ or 2.828

$2\begin{cases} 1: & \text{equation in } k \\ 1: & \text{answer} \end{cases}$

B.

Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.



(a) Find the area of the region R.

(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R

$e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$

1 : Correct limits in an integral in (a), (b), or (c).

(a) Area $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right) dx$

$= 0.590$ or 0.591

$2 \begin{cases} 1: \text{ integrand} \\ 1: \text{ answer} \end{cases}$

(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$

$= 0.55596\pi = 1.746$ or 1.747

$3 \begin{cases} 2: \text{ integrand and constant} \\ \quad < -1 > \text{ each error} \\ 1: \text{ answer} \end{cases}$

(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$

$= 0.461$

$3 \begin{cases} 2: \text{ integrand} \\ \quad < -1 > \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if not of the form} \\ \qquad k\int_c^d (f(x) - g(x))^2\, dx \\ 1: \text{ answer} \end{cases}$

3. Initial Value Problem (No Calculator)

Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2+1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find $f(1.2)$.

Solution	Scoring
(a) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ $\left.\dfrac{dy}{dx}\right\rvert_{\substack{x=1\\y=4}} = \dfrac{3+1}{2\cdot 4} = \dfrac{4}{8} = \dfrac{1}{2}$	**1**: answer
(b) $y - 4 = \dfrac{1}{2}(x-1)$ $f(1.2) - 4 \approx \dfrac{1}{2}(1.2-1)$ $f(1.2) \approx 0.1 + 4 = 4.1$	**2** { 1: equation of tangent line 1: uses equation to approximate $f(1.2)$
(c) $2y\,dy = (3x^2+1)\,dx$ $\int 2y\,dy = \int (3x^2+1)\,dx$ $y^2 = x^3 + x + C$ $4^2 = 1 + 1 + C$ $14 = C$ $y^2 = x^3 + x + 14$ $y = \sqrt{x^3+x+14}$ is branch with point $(1, 4)$ $f(x) = \sqrt{x^3+x+14}$	**5** { 1: separates variables 1: antiderivative of dy term 1: antiderivative of dx term 1: uses $y = 4$ when $x = 1$ to pick one function out of a family of functions 1: solves for y 0/1 if solving a linear equation in y 0/1 if no constant of integration Note: max 0/5 if no separation of variables Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation
(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$	**1**: answer, from student's solution to the given differential equation in (c)

4. Motion (Calculator Allowed)

1. A particle moves along the y–axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.
   (a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?
   (b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$? Why or why not?
   (c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.
   (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $v(1.5) = 1.5\sin(1.5^2) = 1.167$

Up, because $v(1.5) > 0$

1: answer and reason

(b) $a(t) = v'(t) = \sin t^2 + 2t^2\cos t^2$

$a(1.5) = v'(1.5) = -2.048$ or -2.049

No; v is decreasing at 1.5 because $v'(1.5) < 0$

$$\mathbf{2}\begin{cases} 1\text{: } a(1.5) \\ 1\text{: conclusion and reason} \end{cases}$$

(c) $y(t) = \int v(t)\,dt$

$$= \int t\sin t^2\,dt = -\frac{\cos t^2}{2} + C$$

$$y(0) = 3 = -\frac{1}{2} + C \implies C = \frac{7}{2}$$

$$y(t) = -\frac{1}{2}\cos t^2 + \frac{7}{2}$$

$$y(2) = -\frac{1}{2}\cos 4 + \frac{7}{2} = 3.826 \text{ or } 3.827$$

$$\mathbf{3}\begin{cases} 1\text{: } y(t) = \int v(t)\,dt \\ 1\text{: } y(t) = -\frac{1}{2}\cos t^2 + C \\ 1\text{: } y(2) \end{cases}$$

(d) distance $= \int_0^2 |v(t)|\,dt = 1.173$

or

$v(t) = t\sin t^2 = 0$

$t = 0$ or $t = \sqrt{\pi} \approx 1.772$

$y(0) = 3$; $y(\sqrt{\pi}) = 4$; $y(2) = 3.826$ or 3.827

$[y(\sqrt{\pi}) - y(0)] + [y(\sqrt{\pi}) - y(2)]$

$= 1.173$ or 1.174

3
- 1: limits of 0 and 2 on an integral of $v(t)$ or $|v(t)|$
  or
  uses $y(0)$ and $y(2)$ to compute distance
- 1: handles change of direction at student's turning point
- 1: answer
  0/1 if incorrect turning point

5. FTC Application (No Calculator)

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?

(b) How many gallons of water are in the tank at time $t = 3$ minutes?

(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

(d) At what time t, for $0 \le t \le 120$, is the amount of water in the tank a maximum? Justify your answer.

(a) Method 1: $\int_0^3 \sqrt{t+1}\,dt = \frac{2}{3}(t+1)^{3/2}\Big|_0^3 = \frac{14}{3}$

– or –

Method 2: $L(t)$ = gallons leaked in first t minutes

$$\frac{dL}{dt} = \sqrt{t+1}; \quad L(t) = \frac{2}{3}(t+1)^{3/2} + C$$

$$L(0) = 0; \quad C = -\frac{2}{3}$$

$$L(t) = \frac{2}{3}(t+1)^{3/2} - \frac{2}{3}; \quad L(3) = \frac{14}{3}$$

Method 1:

$$3\begin{cases} 2: \text{definite integral} \\ \quad 1: \text{limits} \\ \quad 1: \text{integrand} \\ 1: \text{answer} \end{cases}$$

– or –

Method 2:

$$3\begin{cases} 1: \text{antiderivative with } C \\ 1: \text{solves for } C \text{ using } L(0) = 0 \\ 1: \text{answer} \end{cases}$$

(b) $30 + 8 \cdot 3 - \frac{14}{3} = \frac{148}{3}$

1 : answer

(c) Method 1:

$$A(t) = 30 + \int_0^t \left(8 - \sqrt{x+1}\right)dx$$
$$= 30 + 8t - \int_0^t \sqrt{x+1}\,dx$$

– or –

Method 2:

$$\frac{dA}{dt} = 8 - \sqrt{t+1}$$

$$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + C$$

$$30 = 8(0) - \frac{2}{3}(0+1)^{3/2} + C; \quad C = \frac{92}{3}$$

$$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + \frac{92}{3}$$

Method 1:

$$2\begin{cases} 1: 30 + 8t \\ 1: -\int_0^t \sqrt{x+1}\,dx \end{cases}$$

– or –

Method 2:

$$2\begin{cases} 1: \text{antiderivative with } C \\ 1: \text{answer} \end{cases}$$

(d) $A'(t) = 8 - \sqrt{t+1} = 0$ when $t = 63$

$A'(t)$ is positive for $0 < t < 63$ and negative for $63 < t < 120$. Therefore there is a maximum at $t = 63$.

$$3\begin{cases} 1: \text{sets } A'(t) = 0 \\ 1: \text{solves for } t \\ 1: \text{justification} \end{cases}$$